Reply to 'Comment on Geometric phases for mixed states during cyclic evolutions'

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REPLY TO COMMENT

Reply to 'Comment on Geometric phases for mixed states during cyclic evolutions'

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Abstract
We show that the geometric phase for mixed state during a cyclic evolution suggested in 2004 *J. Phys. A: Math. Gen.* **37** 3699 is $U(1)$ gauge invariant and can be observed by modern techniques.

In the proceeding comment [1], Sjöqvist states that the concept of mixed state geometric phase in cyclic evolution suggested in our recent work [2] is not gauge invariant.

To justify our concept and clarify the issue involved, we have to review the concept of geometric phase. In general, it means that the geometric phase is invariant under $U(1)$ transformation if it is gauge invariant, i.e. the geometric phase is a $U(1)$ gauge invariant. The non-Abelian gauge geometric phase has been suggested only if a set of quantum states remains degenerate as the Hamiltonian varies [3]. The definition of geometric phase of mixed states suggested in our recent work [2] is a $U(1)$ gauge invariant.

Supposing a quantum system with the Hamiltonian $H(t)$, the density operator $\rho(t)$ of this system will undergo the following evolution:

$$\rho(t) = U(t)\rho(0)U^{+}(t) \tag{1}$$

where $U(t) = \mathbf{T}\,\mathrm{e}^{-\mathrm{i}\int_0^t H(t')\mathrm{d}t'/\hbar}$, here $\mathbf{T}$ is the chronological operator. If $[U(\tau), \rho(0)] = 0$, i.e. $\rho(\tau) = \rho(0)$, we say this state undergoes a cyclic evolution with period τ. In [2], we suggested the geometric phase ϕ_g as

$$\phi_g = \phi - \phi_d \tag{2}$$

where

$$\phi = \arg \mathrm{Tr}\,[\rho(0)U(\tau)] \tag{3}$$

is the total phase and

$$\phi_{\mathrm{d}} = -\mathrm{i}\int_0^{\tau} \mathrm{d}t\,\mathrm{Tr}\left[\rho(0)U^{+}(t)\frac{\mathrm{d}U(t)}{\mathrm{d}t}\right] \tag{4}$$

is just the dynamical phase during the cyclic evolution.

The geometric phase can also be expressed as

$$\phi_g = \oint \beta \tag{5}$$

and

$$\beta = \mathrm{iTr}\,[\rho(0)\widetilde{U}^{+}(t)\,\mathrm{d}\widetilde{U}(t)] \tag{6}$$

where $\widetilde{U}(t) = \mathrm{e}^{-\mathrm{i}\phi(t)}U(t)$ such that $\phi(\tau) = \phi$. β is a canonical one-form in the parameter space. The $U(1)$ invariant property of equation (5) has been discussed in detail in [2, 4, 5].

If $U(\tau) = \mathrm{e}^{\mathrm{i}\phi}I$, i.e. $U(\tau)$ is a global cyclic evolution, we proved in [2] that the geometric phase can be expressed as

$$\phi_g = \phi - \sum_k w_k\phi_d^k = \sum_k w_k\phi_g^k. \tag{7}$$

However, if we take the transformation suggested in the comment [1],

$$V(t) = \sum \mathrm{e}^{-\mathrm{i}\alpha_k(t)}\,|\psi_k\rangle\,\langle\psi_k| \tag{8}$$

with $\alpha_k(\tau) - \alpha_k(0) = 2\pi n_k$, then the corresponding unitrary transformation is $U'(\tau)\,|\psi_k\rangle = U(\tau)V(\tau)\,|\psi_k\rangle = \mathrm{e}^{\mathrm{i}(\phi+2\pi n_k)}\,|\psi_k\rangle$. So, $U'(\tau)$ is no more a global cyclic evolution, hence the ϕ_g cannot be expressed as $\sum_k w_k\phi_g'^k$ unless n_k is a constant for any k.

In fact, (8) is not a $U(1)$ transformation when n_k is not a constant for any k. One does not need the definition of geometric phase would be a gauge invariant under such a non-Abelian transformation, since the geometric phase is only a $U(1)$ gauge invariant in general.

In [6], Sjöqvist *et al* proposed a definition of geometric phase for mixed states under the parallel transport condition. We have proven in our work [2] that our definition is similar to their definition when $U(t)$ satisfies the parallel transport condition. Using nuclear magnetic resonance technique, Du *et al* have observed the geometric phase when $U(t)$ satisfies the parallel transport condition [7]. In [2], we have shown that our predictions are similar to what have been observed by Du and his co-workers. Unfortunately, Du and his co-workers have not designed to measure the geometric phase for general cases (for the cases that $U(t)$ is not a parallel transport). In fact, the dynamic phase can be eliminated by the 'spin echo' method if $U(t)$ is not a parallel transport [8], so the geometric phase suggested by equation (2) can be observed experimentally.

In summary, the geometric phase suggested in [2] is $U(1)$ gauge invariant and can be observed by recent techniques. It is improper to demand the geometric phase as a non-Abelian structure in general.

References

[1] Sjöqvist E 2004 *J. Phys. A: Math. Gen.* **37** 7393–4

[2] Fu L-B and Chen J-L 2004 *J. Phys. A: Math. Gen.* **37** 3699

[3] Wilczek F and Zee A 1984 *Phys. Rev. Lett.* **52** 2111
Wilczek F and Zee A 1988 *Phys. Rev.* A **38** 1

[4] Page D N 1987 *Phys. Rev.* A **36** 3479

[5] Pati A K 1995 *Phys. Rev.* A **52** 2576
Pati A K 1995 *J. Phys. A: Math. Gen.* **28** 2087

[6] Sjöqvist E, Pati A K, Ekert A, Anandan J S, Ericsson M, Oi D K L and Vedral V 2000 *Phys. Rev. Lett.* **85** 2845

[7] Du J, Zou P, Shi M, Kwek L C, Pan J-W, Oh C H, Ekert A, Oi D K L and Ericsson M 2003 *Phys. Rev. Lett.* **91** 100403

[8] Bertlmann R A, Durstberger K, Hasegawa Y and Hiesmayr B C 2004 *Phys. Rev.* A **69** 032112